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Low-Fugacity Asymptotic Expansion for Classical Lattice Dipole Gases

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We consider a classical dipole gas in the grand canonical ensemble. We prove that in dimensions greater than or equal to three, and for all temperatures, the free energy and the charges–dipoles correlation functions have an expansion in powers of z, the fugacity of the system, which is asymptotic to all orders. We also give some information about the decay of correlations.

KEY WORDS: Dipole gases; sine-Gordon transformation; charge–charge correlation.

1. INTRODUCTION

In this paper we consider a classical lattice system of particles interacting via a dipole potential in the grand canonical ensemble. In this model, particles occupy sites of a lattice Z^d, and the (unit) dipole moment of each particle takes one of only a finite number of orientations, namely, the $2d$ canonical direction of Z^d. The potential between two particles located at x and y of dipole moment e_x and e_y is given by

$$(2\pi)^{-d}\int_{-\pi}^{\pi} d^dk\left[\exp ik(x-y)\right](\exp ik\cdot e_x-1)(\exp ik\cdot e_y-1)$$
$$\times\left[\sum_{j=1}^{d}(\cos k\cdot e_j-1)\right]^{-1}$$

where e_j, $j=1,\ldots,d$, is the canonical basis of Z^d. The thermodynamic limit of the free energy and of the correlation functions of this system in the grand canonical ensemble is known to exist.[5] Since the pair potential is

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not absolutely integrable, none of the classical methods[10] giving analyticity in z, the fugacity of the system, around $z = 0$ applies in this case. Moreover it is known that the effective dipole potential is not absolutely integrable (absence of screening),[9,6] therefore the cluster expansion used in Ref. 4 for the Coulomb gas does not apply.

Nonetheless, in this paper, we succeed in proving a weaker result than analyticity: in dimensions greater than or equal to 3, the free energy and the correlation functions are asymptotic to all orders in z about $z = 0$. We also have a similar result for correlation functions of charged particles immersed in the dipole gas.

Our results are obtained by first doing a "sine-Gordon transformation"[5,11] which shows the equivalence of our model to a classical system of unbounded spins described by the Hamiltonian

$$-\beta H = -(2\beta)^{-1} \sum_{\langle x,y \rangle} (\phi_x - \phi_y)^2 + z \sum_{\langle x,y \rangle} \cos(\phi_x - \phi_y)$$

where ϕ_x is a random variable uniformly distributed on R, $\beta = T^{-1}$ is the reciprocal temperature, and z is the fugacity. In this language we see that the desired expansion about $z = 0$ is an expansion about a lattice massless Gaussian field. We can therefore apply ideas developed in Refs. 1–3. However, the main difficulty is that the integration by parts formula, which was used in Ref. 1 to generate the expansion, does not appear to be useful to estimate the remainder term after the first n orders have been extracted. Instead we generate the expansion by using the method of "complex translation" introduced in Ref. 7, and used in Refs. 7 and 6 to obtain results about the decay of correlation in $d = 2$. However, we have to introduce a regularization of it to be able to estimate the remainder. To prove our result for charge correlation functions we also make use of the inverse power law decay of the truncated charge–charge two-point function as in Ref. 3. This decay is obtained by applying the FKG inequalities in the form of Ref. 8 and the known inverse power law decay of $\langle \phi_0 \phi_x \rangle$ obtained as in Ref. 2.

Finally the paper is organized as follows: in Section 2 we describe the model and some of its known properties. Section 3 contains the statement and the proof of the main theorem. In Section 4 we prove some technical results used in Section 3. Finally we give some generalizations, and sketch the two-dimensional case.

2. DEFINITION OF THE MODEL

In this paper we consider the following lattice dipole gas: to each point x of the lattice Z^d, $d \geqslant 3$, is associated a unit vector e_x which is an element of $\{e_k\}_{k=1,\ldots,d}$, the usual canonical basis of Z^d ($e_k = \delta_{ik}$), and a variable ϵ_x which takes values ± 1. $\sigma_x = e_x\epsilon_x$ will be viewed as a unit dipole moment of a particle (called a dipole) sitting at x.

Given $f: Z^d \to R$, and $g: Z^d \times Z^d \to R$, we define

$$\nabla^e_x f = f(x) - f(x+e)$$

$$\begin{aligned}\nabla^e_x \nabla^{e'}_y f(x,y) &= \left[f(x,y) - f(x+e,y) \right] \\ &\quad - \left[(f(x,y+e') - f(x+e,y+e') \right]\end{aligned}$$

We are now able to define the *potential between two dipoles* σ_x and σ_y by

$$V(x,y,\sigma_x,\sigma_y) = \epsilon_x\epsilon_y \nabla^{e_x}_x \nabla^{e_y}_y C(x,y)$$

where

$$C(0,x) = (2\pi)^{-d} \int_{-\pi}^{\pi} d^dk \,(\exp ik \cdot x) \left[\sum_{\xi} (1 - \cos k_s) \right]^{-1} \equiv C_{0x}$$

$$k_\xi = k \cdot e_\xi, \qquad \sum_{\xi} \equiv \sum_{e_\alpha,\, \alpha = 1,\ldots,d}$$

The *potential for N dipoles* at position $(x)_N = (x_1 \ldots x_N)$ with dipole moments $(\sigma)_N = (\sigma_1 \ldots \sigma_N)$ is

$$U((x)_N(\sigma)_N) = \sum_{1 \leqslant i \leqslant k \leqslant N} V(x_i, y_k, \sigma_i, \sigma_k)$$

The *grand canonical ensemble partition function* associated to a compact set $\Lambda \subseteq Z^d$ is

$$\Xi(\Lambda,\beta,z) = \sum_{N=0}^{\infty} z^N (N!)^{-1} \sum_{\substack{x_1,\ldots,x_N \in \Lambda \\ \sigma_1 \ldots \sigma_N}} \exp(-\beta U((x)_N,(\sigma)_N))$$

where z is the *activity* or *fugacity* and $\beta = T^{-1}$ is the inverse temperature.

The *correlation functions* $\rho_\Lambda((x^0)_M(\sigma^0)_M,\beta,z)$ equal

$$\sum_{N=0}^{\infty} z^{N+M} (N!)^{-1} \sum_{\substack{x_1,\ldots,x_N \in \Lambda \\ \sigma_1 \ldots \sigma_N}} \exp\left(-\beta U\left((x^0)_M,(\sigma^0)_M,(x)_N,(\sigma)_N\right)\right)$$

We also introduce correlation functions of charged particles immersed in the dipole gas. A particle of charge $q \in Z$ located at a point $x \in Z^d$ will be denoted by (x,q). The charge–charge potential $V_1(x,q,x',q') = qq'C(xx')$, and the charge–dipole potential $V_2(x,q,y,\sigma) = q\epsilon\nabla^{e_y}_y C(x,y)$.

The *general correlation function* of L charges $(x^0)_L = (x_1^0 \dots x_L^0)$, $(q^0)_L = (q_1 \dots q_L)$ and M dipoles $(y^0)_M, (\sigma^0)_M$ is

$$\rho_\Lambda\big((x^0)_L(q)_L; (y^0)_M(\sigma^0)_M; \beta z\big)$$
$$= \sum_{N=0}^{\infty} z^{M+L+N}(N!)^{-1}$$
$$\times \sum_{\substack{x_1 \dots x_N \in \Lambda \\ \sigma_1 \dots \sigma_N}} \exp\big(-\beta U\big((x^0)_L, (q)_L, (y^0)_M, (\sigma^0)_M, (x)_N(\sigma)_N\big)\big)$$

where

$$U((x)_L, (q)_L, (y)_M, (\sigma)_M) = \sum_{1 \leqslant i, j \leqslant L} V_1(x_i q_i x_j q_j) + \sum_{\substack{1 \leqslant j \leqslant L \\ 1 \leqslant j \leqslant M}} V_2(x_i q_i y_j \sigma_j)$$
$$+ \sum_{1 \leqslant i, j \leqslant M} V(y_i \sigma_i y_j \sigma_j)$$

2.1. The sine-Gordon Transformation[5,11]

Let $d\mu_\beta$ be the usual Gaussian measure associated to the lattice massless free field of covariance $\beta C(0,x)$.

Proposition 1.[5,11]

(a)
$$\Xi_\Lambda = Z_\Lambda = \int d\mu_\beta \left[\exp z \sum_{\substack{x \in \Lambda \\ \xi}} (\cos \nabla_x^\xi \phi)\right]$$

(b) If

$$\langle \cdot \rangle_{\Lambda,z} = Z_\Lambda^{-1} \int \cdot \, d\mu_\beta \left[\exp z \sum_{\substack{x \in \Lambda \\ \xi}} (\cos \nabla_x^\xi \phi)\right]$$

$$\rho_\Lambda((x)_L(q)_L; (y)_M(\sigma)_M, \beta, z) = z^{M+L} \left\langle \prod_{k=1}^{L} \exp i q_k \phi(x_k) \times \prod_{l=1}^{M} \exp i \epsilon_l \nabla_{x_l}^{e_l} \phi \right\rangle_{\Lambda,z}$$

2.2. Correlation Inequalities

Let $m: Z^d \to \mathbb{R}$ be a function of finite support, and define $\phi(m) = \sum_i \phi_i m(i)$.

Proposition 2.[5]

(a) $\langle \cos\phi(m)\cos\phi(n)\rangle_{\Lambda,z} - \langle \cos\phi(m)\rangle_{\Lambda,z}\langle \cos\phi(n)\rangle_{\Lambda,z} \geqslant 0$

(b) $\langle [\phi(m)]^2 \cos\phi(n)\rangle_{\Lambda,z} - \langle [\phi(m)]^2\rangle_{\Lambda,z}\langle \cos\phi(n)\rangle_{\Lambda,z} \leqslant 0$

The pressure $p_\Lambda(\beta, z) = |\Lambda|^{-1} \log Z_\Lambda$.

Proposition 3.[5]

(a) $$\lim_{\Lambda\to\infty} p_\Lambda(\beta, z) = p(\beta, z) \text{ exists.}$$

(b) $$\lim_{\Lambda\to\infty} \rho_\Lambda((x)_L, (q)_L; (y)_M, (\sigma)_M; \beta, z)$$
$$= \rho((x)_L (q)_L; (y)_M, (\sigma)_M : \beta, z) \text{ exists.}$$

(c) $$\lim_{\Lambda\to\infty} \langle \phi(x)\phi(y)\rangle_{\Lambda,z} = \langle \phi(y)\phi(y)\rangle \text{ exists.}$$

Remarks. (1) By the usual arguments, Proposition 2(a) and (b) imply Proposition 3(b) and (c). Moreover

$$\langle (\phi(n))^2\rangle_z \leqslant \langle (\phi(n))^2\rangle_{z=0} < \infty \tag{1}$$

(2) Brascamp–Lieb inequalities[13] can be applied for $z < \beta^{-1}$ where they give the estimate $\langle \exp|\phi_0|\rangle_z < \infty$.

In what follows since β does not play any significant role we shall assume $\beta = 1$.

2.3. Decay of Correlations

Proposition 4.

(a) $$0 \leqslant \langle \phi_0 \phi_x \rangle_z \leqslant c|x|^{-1} \ln|x| \qquad \text{for } d = 3$$

(b) $$0 \leqslant \langle \phi_0 \phi_x \rangle_z \leqslant c|x|^{-1} \qquad \text{for } d \geqslant 4$$

(c) $$|\langle \nabla_0^e \phi \nabla_x^{e'} \phi \rangle_z| \leqslant c|x|^{-1} \qquad \text{for all } d$$

This proposition is a direct consequence of Theorem 2 of Ref. 2 because of the momentum space bound implied by (1) and because of the monotonicity property: $\langle \phi_0 \phi_x \rangle_z \leqslant \langle \phi_0 \phi_{x+1} \rangle_z$, where x and $x + 1$ are along a coordinate axis. This last property follows from reflection positivity and translation invariance; the existence of a transfer matrix needs not to be assumed.

The system satisfies FKG inequalities in the ϕ variables; this is an easy consequence of the Battle–Rosen condition,[14] as has been pointed out to us by J. Fröhlich. Therefore Newman's theorem[8] can be applied to show

$$\left|\left\langle \exp i \sum_{l=1}^{m} q_l \phi_l \right\rangle - \prod_{l=1}^{m} \langle \exp i q_l \phi_l \rangle \right| \leqslant \tfrac{1}{2} \sum_{l \neq n} |q_l|\,|q_n| \langle \phi_l \phi_n \rangle$$

In particular we have the following decay of the charge–charge truncated correlation function.

Proposition 5.

$$|\rho(0, q_0, x, q_x) - \rho(0, q_0)\rho(0, q_x)| \leqslant c|q_0 q_1|\,|x|^{-1}\ln|x|$$

N.B. In the whole paper C will stand for a positive constant which can take different values at different places.

3. THE MAIN RESULT

Theorem. In dimension greater than or equal to three, and for all temperatures the free energy $p(z, \beta)$ and the correlation functions $\rho((x)_L, (q)_L; (y)_M, (\sigma)_M; z, \beta)$ have about $z = 0$ an asymptotic expansion to *all order in* z. (The first term in the expansion of the correlation function is of order z^{L+M}.)

Before starting the proof of the theorem we explain how to use the method of "*complex translation*"[7] to generate the expansion. Let us consider the expansion of

$$\langle \exp i \nabla_0^e \phi \rangle_{\Lambda, z} = Z_\Lambda^{-1} \int (\exp i \nabla_0^e \phi) \exp\left(z \sum_{\substack{x \in \Lambda \\ \xi}} (\cos \nabla_x^\xi \phi) \right) d\mu(\phi) \qquad (2)$$

Doing the complex translation $\phi_x \to \phi_x + i a_x$, with $a_x = \nabla_0 C_{0x}$ in the functional integral of the numerator of (2) we obtain

$$\langle \exp i \nabla_0^e \phi \rangle = Z_\Lambda^{-1} \int (\exp - \nabla_0^e a)(\exp i \nabla_0^e \phi) \left(\exp z \sum_{\substack{x \in \Lambda \\ \xi}} \cos\left(\nabla_x^\xi \phi + i \nabla_x^\xi a\right) \right)$$
$$\times \left[\exp \frac{1}{2} \sum_{x, \xi} \left(\nabla_x^\xi a\right)^2 \right] \left[\exp - i \sum_{x, \xi} \nabla_x^\xi a \nabla_x^\xi \phi \right] d\mu(\phi) \qquad (3)$$

Now $\sum_{x,\xi} (\nabla_x^\xi a)^2 = \sum_x a_x(-\Delta_x) a$, where Δ_x is the finite difference Laplacian. With our choice of a_x and using $-\Delta(\nabla_0 C_{0x}) = \delta_{0x} - \delta_{1x}$, we have

$$\sum_{x, \xi} \left(\nabla_x^\xi a\right)^2 = \langle (\nabla_0 \phi)^2 \rangle_G$$

as the Gaussian expectation value. Similarly

$$\sum_{x,\xi} \nabla_x^\xi a \nabla_x^\xi \phi = \phi_0 - \phi_1$$

Therefore,

$$(3) = Z_\Lambda^{-1}\left[\exp - \frac{1}{2}\langle(\nabla_0\phi)^2\rangle_G\right]\int\left[\exp z \sum_{\substack{x\in\Lambda \\ \xi}} \cos(\nabla_x^\xi + i\nabla_x^\xi a)\right] d\mu(\phi) \quad (4)$$

One should really derive this formula by doing complex translation on the Gaussian measure restricted to a finite set $\Lambda_0 \supset \Lambda$ and then take $\Lambda_0 \to \infty$ to obtain the result. The right-hand side of (4) may be written as a zeroth-order term, A_0, plus remainder, $R(z)$:

$$\begin{aligned}
(3) &= \exp - \tfrac{1}{2}\langle(\nabla_0\phi)^2\rangle_0 \\
&\quad + A_0\left\langle \exp\left\{ z \sum_{\substack{x\in\Lambda \\ \xi}} \left[\cos(\nabla_x^\xi\phi + i\nabla_x^\xi a) - \cos\nabla_x^\xi\phi\right]\right\} - 1\right\rangle \\
&= A_0 + A_0\left\langle\left\{\exp\left[z \sum_{\substack{x\in\Lambda \\ \xi}} \cos\nabla_x^\xi\phi(\cosh\nabla_x^\xi a - 1)\right] - 1\right\}\right. \\
&\quad \left.\times \exp\left(i \sum_{\substack{x\in\Lambda \\ \xi}} \sin\nabla_x^\xi\phi \sinh\nabla_x^\xi a\right)\right\rangle \\
&\quad + A_0\left\langle \exp\left(iz \sum_{x,\xi} \sin\nabla_x^\xi\phi\sinh\nabla_x^\xi a\right) - 1\right\rangle \qquad (5)
\end{aligned}$$

However, as we shall see next, the remainder $R(z)$ is not easily shown to be small with z. As in Ref. 1 we shall not expand about a massless theory but rather around a theory of small mass $m(z) = \exp - (\ln z)^2$. This will be done by translating ϕ_x by ia_x, where a_x is a function of the *massive* covariance C_{0x}^m of the lattice Gaussian field of mass $m = m(z)$. For instance $a_x = \nabla_0 C_{0x}^m$.

Proof of the Theorem. (1) The result for the *free energy* easily follows from the result on the correlation functions and the formula

$$p(z) = p(0) + \int_0^z \sum_\xi \langle\cos\nabla_0^\xi\phi\rangle_{z'}\, dz'$$

We prove the result for the correlation functions by first considering $L = 0$.

(2) *Dipole correlation functions.* We start by estimating the remainder of $\langle \exp i\nabla_0^e\phi\rangle$ after extraction of the zeroth-order perturbation theory. Applying (3) with $a_x = \nabla_0^e C_{0x}^m$ and using $-\Delta C_{0x}^m = \delta_{0x} - m^2 C_{0x}^m$, we get

$$\langle \exp i\nabla_0^e\phi\rangle = \exp - \tfrac{1}{2}\langle(\nabla_0\phi)^2\rangle_{G,m} Z_\Lambda^{-1} \int \left[\exp z \sum_{x,\xi} \cos\left(\nabla_x^\xi\phi + i\nabla_x^s a\right)\right]$$
$$\times \left[\exp im^2 \sum_j \left(\nabla_0^e C_{0j}\phi_j\right)\right] \exp\left[-\frac{1}{2} m^2 \sum_j \left(\nabla_0^e C_{0j}\right)^2\right] d\mu(\phi) \qquad (6)$$

Denote the massive Gaussian expectation value $\exp - \frac{1}{2}\langle(\nabla_0^e\phi)^2\rangle_{G,m}$ by A_0^m. Then (6) becomes

$$A_0^m\langle \exp\left\{ z\sum_{x,\xi}\left[\cos\left(\nabla_x^\xi\phi + i\nabla_x^\xi a\right) - \cos\nabla_x^\xi\phi\right]\right\}$$
$$\times \exp\left(im^2\sum_x \nabla_0^e C_{0x}\phi_x\right)\exp\left[-\frac{1}{2}m^2\sum_x\left(\nabla_0^e C_{0x}\right)^2\right] - 1\rangle \qquad (7)$$

Calling the three exponentials in (7), respectively, E_1, E_2, E_3, one has $\langle \exp i\nabla_0^e\phi\rangle = A_0^m + R_1(z) + R_2(m) + R_3(m)$, with $R_1(z) = \langle E_1 - 1\rangle$, $R_2(m) = \langle E_1E_2(E_3 - 1)\rangle$, $R_3(m) = \langle E_1(E_2 - 1)\rangle$. In the next section we shall show $|A_0^m - A_0^{m=0}| \leqslant m$ const. Therefore the remainder $R(z)$ has been split into four parts with $R_1(z)$, the z term, similar to $R(z)$ except that it contains the massive covariance. $R_2(m)$ and $R_3(m)$ will be called the mass terms. They correspond to the mass terms introduced in the regularized integration by parts formula of Ref. 1. We shall show that they may be ignored in the expansion because they are proportional to $\exp - (\ln z)^2$.

Estimate of the remainder

Lemma.

$$\sum_{x,\xi}|\cosh\nabla_x^\xi a - 1| \leqslant C \quad \text{and} \quad \sum_{x,\xi}|\sinh\nabla_x^\xi a| \leqslant C\ln m$$

The proof of this lemma follows from the estimates

$$\begin{aligned} \cosh x - 1 &\leqslant x^2 \qquad |x| < 1 \\ \sinh|x| &\leqslant 2|x| \qquad |x| < 1 \end{aligned} \qquad (8)$$

and

$$\sum_x \left(\nabla_0^e\nabla_x^{e'} C_{0x}\right)^2 < \infty$$

$$\sum_x |\nabla_0^e\nabla_x^{e'} C_{0x}| < C\ln m \qquad \text{see Ref. 1}$$

We are now ready to estimate the different parts of the remainder:

$$\text{(a)} \qquad R_1(z) = \Big\langle \Big\{ \exp\Big[z \sum_{x,\xi} (\cos \nabla_x^\xi \phi)(\cosh \nabla_x^\xi a - 1) \Big] - 1 \Big\}$$

$$\times \exp\Big(iz \sum_{x,\xi} \sinh \nabla_x^\xi a \sin \nabla_x^\xi \phi \Big) \Big\rangle$$

$$+ \Big\langle \cos\Big(z \sum_{x,\xi} \sin \nabla_x^\xi \phi \sinh \nabla_x^\xi a \Big) - 1 \Big\rangle$$

$$+ i \Big\langle \sin\Big(z \sum_{x,\xi} \sin \nabla_x^\xi \phi \sinh \nabla_x^\xi a \Big) \Big\rangle \tag{9}$$

Applying Taylor's theorem together with the lemma one gets $|R_1(z)| \leqslant Cz \exp Cz + z^2 C(\ln m)^2$. If m were not z dependent, this estimate would blow up in the limit $m \downarrow 0$. This is why we had to introduce a z-dependent mass and could not estimate directly the remainder $R(z)$ in the form of (7).

$$\text{(b)} \qquad R_2(m) \leqslant \langle |E_1| \, |E_2| \, |E_3 - 1| \rangle.$$

$|E_1| \leqslant C$ by estimates similar to those done in case (a). $|E_2| \leqslant 1$ and $|E_3 - 1| \leqslant Cm^2$ by the lemma and Taylor's theorem. Therefore $R_2(m) \leqslant Cm^2$.

$$\text{(c)} \qquad R_3(m) \leqslant \langle |E_1| \, |E_2 - 1| \rangle$$

$$\langle |E_2 - 1|^2 \rangle = \Big\langle 2 - 2\cos\Big(m^2 \sum_x \nabla_0^e C_{0x} \phi_x \Big) \Big\rangle$$

Again by Taylor's theorem

$$\begin{aligned}
\langle |E_2 - 1|^2 \rangle^{1/2} &\leqslant C \Big[m^2 \sum_{x,y} \nabla_0^e C_{0x} \nabla_0^e C_{0y} \langle \phi_x \phi_y \rangle \Big]^{1/2} \\
&\leqslant C \Big[\langle \phi_0^2 \rangle m^4 \Big(\sum_x \nabla_0^e C_{0x} \Big)^2 \Big]^{1/2} \\
&\leqslant Cm \langle \phi_0^2 \rangle^{1/2} \qquad \text{by Ref. 1} \\
&\leqslant Cm \qquad \text{by Remark (1)}
\end{aligned}$$

The higher-order terms are obtained by expanding $R_1(z)$. For instance terms of second order are given by

$$z \sum_{x,\xi} (\cosh \nabla_x^\xi a - 1) \langle \cos \nabla_x^\xi \phi \rangle_1 + \frac{z^2}{2} \sum_{\substack{x,\xi \\ y,\eta}} (\cosh \nabla_x^\xi a - 1)(\cosh \nabla_y^\eta a - 1)$$

$$\times \langle \cos \nabla_x^\xi \phi \cos \nabla_y^\eta \phi \rangle_0 - \frac{z^2}{2} \sum_{\substack{x,\xi \\ y,\eta}} \sinh \nabla_x^\xi a \sinh \nabla_y^\eta a \, \langle \sin \nabla_x^\xi \phi \sin \nabla_y^\eta \phi \rangle$$

where $\langle\,\rangle_i$ means ith order in perturbation theory. This easily follows from Taylor's theorem and the lemma. The remainder given by Taylor's theorem is bounded by $Cz^3 + z^4(\ln m)^4$. Now

$$\langle \sin \nabla_x^\xi \phi \rangle_1 = \exp\left\{ -\langle (\nabla_0^e \phi)^2 \rangle_{0,m} z \left[\sum_{x,\xi} (\cosh \nabla_x^\xi a - 1) \right]^2 \right\}$$

by applying the preceding procedure twice.

$$\langle \sin \nabla_x^\xi \phi \sin \nabla_y^\eta \rangle_0 = \sum_{\epsilon_x, \epsilon_y = \pm 1} \langle \exp\left(i\epsilon_x \nabla_x^\xi \phi + i\epsilon_y \nabla_y^\eta \phi \right) \rangle_0$$

The result follows by translating $\phi_u \to \phi_u + ib_u$ with $b_u = \epsilon_x \nabla_x^\xi C x_u + \epsilon_y \nabla_y^\eta C y_u$. The estimates of R_1, R_2, R_3 are done as before. There are, however, extra $\ln m$ factors coming from $\sum_{x,\xi,y,\eta} |\sinh \nabla_x^\xi b|\, |\sinh \nabla_y^\eta b|$. It is easy to generalize the preceding procedure to higher order of general correlation functions (see Section 4).

Each time we apply the complex translation formula three kinds of terms are produced:

(a) Purely Gaussian terms [e.g., $\exp - \frac{1}{2} \langle (\nabla_0 \phi)^2 \rangle_{G,m}$].
(b) z terms which are contained in $R_1(z)$ and are small with respect to the Gaussian terms.
(c) Mass terms which are proportional to $\exp - (\ln z)^2$ and therefore disappear from the expansion.

So in general we shall obtain

$$\rho((x)_N, (\sigma)_N, \beta, z) = z^N \left[A_0(m) + A_1(m) z + \cdots + A_k(m) z^k + O(z^{k+\epsilon}) \right]$$

This proves the theorem because in Section 4 we shall prove

$$\forall l, \ |A_l(m) - A_l(m = 0)| \leqslant Cm = C \exp - (\ln z)^2$$

(3) *Charge correlation functions*. For charge correlations, we apply essentially the method described in Ref. 3. That is, if we want the expansion of $\langle \exp i\phi_0 \rangle = \langle \cos \phi_0 \rangle$, we first write $\langle \cos \phi_0 \rangle^2 = \cos(\phi_0 - \phi_x) - [\langle \cos(\phi_0 - \phi_x) \rangle - \langle \cos \phi_0 \rangle^2]$. By Proposition 5 the second term is bounded by $C|x|^{-1} \ln|x|$. Therefore if we want an expansion up to order n we choose $|x| = z^{-(n+1)}$ so that $|x|^{-1} \ln|x| \leqslant |\ln z| z^{n+1}(n+1)$, i.e., it will be part of the remainder of the expansion. We now perform an expansion in z up to order n of $\langle \cos(\phi_0 - \phi_x) \rangle_z$ the dependence in z of which comes from the measure and from $x = z^{-(n+1)}$. As before the expansion of $\langle \exp i(\phi_0 - \phi_x) \rangle$ is generated by complex translation. Let us consider the remainder after the first translation $\phi_u \to \phi_u + i(C_{0u} - C_{xu}) = \phi_u + ia_u$.

(a) *The mass terms*. As

$$|\nabla_u (C_{0u} - C_{xu})| \leqslant |x| \nabla_0 \nabla_u C_{0u}, \qquad |C_{0u} - C_{x0}| \leqslant |x|\, |\nabla_0 C_{0u}| \tag{10}$$

the estimates we had in (2) for the mass terms are multiplied by a factor $|x| = z^{-(n+1)}$. They remain negligible in the expansion.

(b) *The z terms* $R_1(z)$. As before we Taylor expand the functions in $R_1(z)$ up to order $n+1$ in z. A typical coefficient of this expansion is, for instance,

$$z^{2k}(2k!)^{-1} \sum_{x_1 \cdots x_k} \sinh \nabla_{x_1} a \cdots \sinh \nabla_{x_k} a \langle \sin \nabla_{x_1} \phi \cdots \sin \nabla_{x_k} \phi \rangle$$

(in which we have dropped the direction of the gradients to make notation simpler). The expectation value $\langle \sin \nabla_{x_1} \phi \cdots \sin \nabla_{x_k} \phi \rangle$ is reexpanded up to order z^α with $\alpha = 2k(n+1) + n + 1$. By the estimates of case 2(a), the new z terms produced will be bounded by

$$Cz^\alpha z^{-2k(n+1)} = Cz^{n+1}(\ln m)^\alpha$$

which is small compared to z^n.

After this first step, the Gaussian terms produced are still z dependent since $x = z^{-(n-1)}$. We reexpand them in z as in Ref. 3. However, we have to use the multiangle formula for sinh and cosh (13) to write them as Gaussian expectation values involving only ϕ_0 or ϕ_x (which do not depend upon x by translation invariance), and Gaussian expectation values mixing ϕ_0 and ϕ_x. These last terms are negligible in the expansion because they are of order $z^{n+1/2}$. This is proven as in Ref. 3 by using Weinberg's theorem[12] instead of the explicit computation. We finally get the result after applying Proposition 6. The case of more general charge correlation functions is done using reflection positivity as in Ref. 3. ∎

4. GAUSSIAN ESTIMATES

In this section we first show that the coefficients of the perturbation expansion in z of the correlation functions are finite. We then estimate the difference between coefficients computed in a massive and a massless Gaussian theory. These two results are used in the proof of the theorem.

In this section we drop the direction symbols in the ∇_x's since they play no significant role.

4.1. Construction of Graphs and Finiteness of Perturbation Theory

Let us consider the expansion of

$$\langle \cos \nabla_0 \phi \cdots \cos \nabla_n \phi \sin \nabla_{n+1} \phi \cdots \sin \nabla_{n+2p} \phi \rangle \tag{11}$$

where

$$2 \cos \nabla\phi = \sum_{\epsilon = \pm 1} (\exp i\epsilon \nabla\phi), \qquad 2i \sin \nabla\phi = \sum_{\epsilon = \pm 1} \epsilon(\exp i\epsilon \nabla\phi)$$

the zeroth-order term is obtained by $\phi_x \to \phi_x + ib_x$ with

$$b_x = \epsilon_0 \nabla_0 C_{0x} + \cdots + \epsilon_{n+2p} \nabla_{n+2p} C_{n+2px}; \ \epsilon = (\epsilon_1, \ldots, \epsilon_{n+2p})$$

$$\begin{aligned}(11) = \sum_\epsilon \epsilon_{n+1} \cdots \epsilon_{n+2p} \prod_{(x,y)\in A} &(\sinh \epsilon_x \epsilon_y \nabla_x \nabla_y C_{xy} + \cosh \nabla_x \nabla_y C_{xy}) \\ + \epsilon_{n+1} \cdots \epsilon_{n+2p}\Big[&\Big\langle \Big\{ \exp\Big[z \sum_x (\cosh \nabla_x b - 1) \cos \nabla_x \phi \Big] - 1 \Big\} \\ &\times \exp\Big(iz \sum_x \sinh \nabla_x b \sin \nabla_x \phi \Big) \Big\rangle \\ &+ \Big\langle \exp\Big(iz \sum_x \sinh \nabla_x b \sin \nabla_x \phi \Big) - 1 \Big\rangle \Big] \\ &(+ \text{ terms proportional to } m)\end{aligned}$$

($A = \{0 \ldots n + 2p\}$). A term of order k in the expansion of (11) is given by

$$\begin{aligned} z^s \sum_\epsilon \epsilon_{n+1} \cdots \epsilon_{n+2p} &(\cosh \nabla_{x_1} b - 1) \cdots (\cosh \nabla_{x_l} b - 1) \\ &\times \sinh \nabla_{x_{l+1}} b \cdots \sinh \nabla_{x_s} \\ &\times b \langle \cos \nabla_{x_1} \phi \cdots \cos \nabla_{x_l} \phi \sin \nabla_{x_{l+1}} \phi \cdots \sin \nabla_{x_s} \phi \rangle_t \end{aligned} \tag{12}$$

$k = s + t$ and $t < k$. We apply the multiangle formulas

$$\begin{aligned} \cosh(x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y \\ \sinh(x \pm y) &= \sinh x \cosh y \pm \cosh x \sinh y \end{aligned} \tag{13}$$

successively to $\cosh \nabla b$ and $\sinh \nabla b$, and we write

$$\begin{aligned} (\cosh \nabla_{x_j} b - 1) = \Big(\prod_{k=0 \cdots n+2p} &\cosh \nabla_{x_j} \nabla_k C_{x_j k} - 1 \Big) \\ &+ \text{other terms (involving sinh and cosh)} \end{aligned}$$

The different terms produced can be represented graphically; the propagators $\sinh \nabla_x \nabla_y C_{xy}$ and $(\cosh \nabla_x \nabla_y C_{xy} - 1)$ are respectively represented by a line and a dashed line between x and y. ($\cosh \nabla_x \nabla_y C_{xy}$ will not be considered since it does not affect convergence.) $1 \ldots n + 2p$ are called external points and $x_1 \ldots x_s$ are called internal points. We shall now establish the restrictions imposed on such graphs.

(*a*) *Case of the sinh lines.* Applying the multiangle formula to $\cosh \nabla_{x_j} b$, we find that each term contains an even number of sinh lines going out of x_j ($j = 1 \ldots l$) (essentially because cosh is an even function). However, there will be an odd number of lines going out of x_{k_j}, $j = l + 1 \ldots s$ (because sinh is odd). All these lines join internal points to external points.

After summation on ϵ, graphs will cancel and only those graphs with an even number of lines arriving at $0 \ldots n$ and an odd number of lines arriving at $n+1 \ldots n+2p$ will survive.

(*b*) *Case of the* (*cosh* $-$ 1) *lines*. The terms $(\prod_{k=0 \ldots n+2p} \cosh \nabla_{x_j} \nabla_k C_{x_{jk}} - 1)D$ (D is a function of sinh and cosh) can be written as a sum of terms $\sum(\cosh \nabla_{x_j} \nabla_k C_{x_{jk}} - 1)BD$ (B is a function of cosh) by means of the formula $xy - 1 = \frac{1}{2}[(x-1)(y+1) + (y-1)(x+1)]$. In other words a (cosh $-$ 1) line is produced between x_j and an external point. Again after summation on ϵ no such line arrives at the external points $n+1 \ldots n+2p$.

These considerations yield the following as the conditions which such graphs must satisfy. A graph of order k has k internal points. At each internal point an even number of sinh lines terminate, and an arbitrary number of (cosh $-$ 1) lines. Moreover, each connected component of the graph must contain at least an external point.

To each graph one associates a number by summing over all the internal variables. To prove the convergence of these sums *uniformly in m*, we apply Weinberg's theorem[12] because either a point belongs to a (cosh $-$ 1) line, or it belongs to an *even* number of sinh lines. Similar rules can be obtained to prove the convergence of the graph produced in the expansion of charge correlation functions.

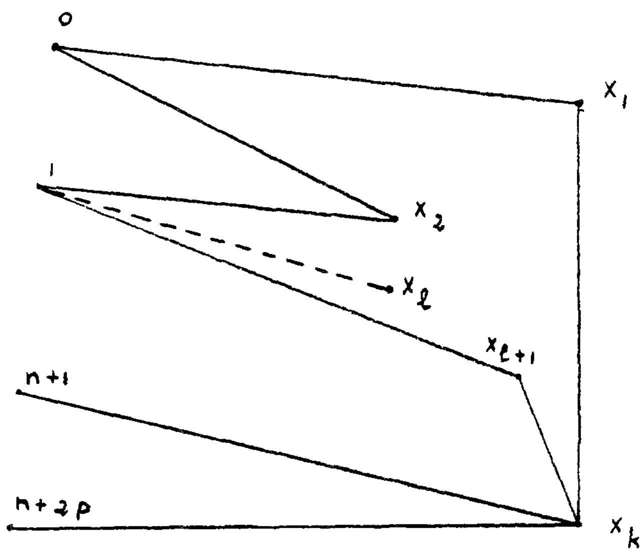

Fig. 1. A graph of order k.

4.2. Differences of Coefficients

Suppose the asymptotic expansion of a correlation function is given by $a_0(m) + a_1(m)z + \cdots + a_k(m)z^k + O(z^{k+\epsilon})$ where the $a_i(m)$ are computed with a covariance of mass m. Then we have the following proposition.

Proposition 6.

$$\forall i, |a_i(m) - a_i(0)| \leqslant m(\ln m)^i$$

Proof. Since we cannot compute explicitly the graphs in Fourier space, the proof of this proposition differs slightly from the one given in Ref. 1.

(*a*) *Case of dipole correlations.* As in Ref. 1 it is sufficient to estimate $\Delta_m G = G_m - G_0$, where G_m is a graph of order k and G_0 is the same graph but the massive propagator $\sinh \nabla_0 \nabla_x C_{0x}^m$ is replaced in one line by the massless one [the case of $(\cosh - 1)$ lines is easier]:

$$|\sinh \nabla_0 \nabla_x C_{0x}^m - \sinh \nabla_0 \nabla_x C_{0x}^{m=0}| \leqslant |\nabla_0 \nabla_x C_{0x}^m - \nabla_0 \nabla_x C_{0x}^{m=0}| C$$

$$\leqslant m^2 C \sum_x |C_{xy}| \, |\nabla_0 \nabla_y C_{0y}| \equiv m^2 f(x)$$

The new graph G, defined by replacing (in the chosen line) the massive propagator by $f(x)$, is still convergent. This follows from Weinberg's theorem.[12] Therefore $|\Delta_m G| \leqslant Cm^2$.

(*b*) The coefficients arising in the expansion of a charge correlation function contain two kinds of sinh propagator:

(1) A propagator connecting an external point to an internal point $\sinh \nabla_0 C_{0x}$.

(2) A propagator connecting internal points: $\sinh \nabla_0 \nabla_x C_{0x}$. For a $\sinh \nabla_0 C_{0x}$ line,

$$|\sinh \nabla_0 C_{0x}^m - \sinh \nabla_0 C_{0x}| \leqslant Cm^2 \sum_x C_{xy} |\nabla_0 C_{0y}| \equiv m^2 g(y)$$

By applying Weinberg's theorem as above for a graph of order k one gets $|\Delta_m G| \leqslant C(\ln m)^k m$. In the case of a $(\sinh \nabla_0 \nabla_x C_{0x})$ line $|\Delta_m G| \leqslant C(\ln m)^k m^2$.

N.B. We have not considered the self-energy graphs connecting a point to itself, e.g., $\exp[-\frac{1}{2}\langle(\nabla_0 \phi)^2\rangle_G]$. They do not present any difficulty; one can resume them by Wick ordering the interaction, which is equivalent to multiplying z by a constant.

Remarks. (1) The theorem is also true for the dipole correlation functions in the case of a system of arbitrary length dipoles, provided the maximal length of the dipoles is finite. In the sine-Gordon representation this corresponds to an interaction

$$z \sum_{\substack{x, e_\xi \\ l=1 \ldots k<\infty}} \cos\left[\phi(x) - \phi(x + le_\xi)\right]$$

All the estimates we had for the nearest-neighbors case are multiplied by a finite constant.

However, in this more general model we cannot prove the existence of an asymptotic expansion for the charge correlation functions. This is because we do not have reflection positivity for non-nearest-neighbor interactions.

(2) In two dimensions, the model is well defined and so is the sine-Gordon transformation. In particular the dipole correlation functions make sense. We now sketch one way of proving that they are asymptotic to all orders in z. We essentially use the method of Section IV of Ref. 3. Namely, as remarked by J. Bricmont, Proposition 2 implies

$$\langle \cos\phi(n)\rangle_\Lambda \leqslant \langle \cos\phi(n)\rangle \leqslant \langle \cos\phi(n)\rangle_m$$

where

(1) $\phi(n) = \sum_i \phi_i n_i, \qquad n_i \in Z, \quad \text{and} \quad \sum n_i = 0.$

(2) $\langle \cos\phi(n)\rangle_{\Lambda,m} = Z(\Lambda, m)^{-1} \int \cos\phi(n) \exp\left(z \sum_{\substack{x\in\Lambda \\ \xi}} \cos \nabla_x^\xi \phi \right) d\mu_{\beta,m}$

[$d\mu_{\beta,m}$ is the usual Gaussian measure of covariance βC_{0x}^m, and $Z(\Lambda, m)$ is a normalization factor such that $\langle 1\rangle_{\Lambda,m} = 1$]

(3) $\langle \cdot \rangle_\Lambda = \langle \cdot \rangle_{\Lambda, m=0}, \langle \cdot \rangle_m = \lim_{\Lambda\to\infty} \langle \cdot \rangle_{\Lambda,m}$

Now choosing the radius of the box Λ, $R(\Lambda) = \exp - (\sqrt{z})^{-1}$ and the mass, $m(z) = \exp - (\ln z)^2$, we can expand $\langle \cos\phi(n)\rangle_\Lambda$ and $\langle \cos\phi(n)\rangle_m$ in z (and estimate the remainder) using complex translations involving, respectively, $C(0,x)$ and $C^m(0,x)$. This gives

$$\langle \cos\phi(n)\rangle_\Lambda = \sum_{i=1}^{k} a_i^\Lambda z^i + O(z^{k+\epsilon})$$

$$\langle \cos\phi(n)\rangle_m = \sum_{i=1}^{k} a_i(m) z^i + O(z^{k+\epsilon})$$

$a_i(m)$ are as in Proposition 6; a_i^Λ are coefficients computed with propagator C_{0x}, but with all lattice sums restricted to Λ (for example,

$\sum_{x,y \in \Lambda} \sinh \nabla_0 \nabla_x C_{0x} \sinh \nabla_x \nabla_y C_{xy} \sinh \nabla_y \nabla_0 C_{0y}$). It is expected that all a_i^Λ converge to their limit (as $\Lambda \to \infty$) at least as fast as $R(\Lambda)^{-\alpha}$ ($\alpha > 0$), but we have not checked this in detail. Therefore up to exponentially small terms in z, $a_i(m)$ and a_i^Λ coincide, which proves the result.

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